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RESEARCH ARTICLE

η-Simple semigroups without zero and η^*-simple semigroups with a least non-zero idempotent

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Abstract A semigroup S is called *η-simple* if S has no semilattice congruences except $S \times S$. Tamura in (Semigroup Forum 24:77–82, 1982) studied η-simple semigroups with a unique idempotent. In the present paper we consider a more general situation, that is, we investigate η-simple semigroups (without zero) with a least idempotent. Moreover, we study η^*-simple semigroups with zero which contain a least non-zero idempotent.

Keywords η-Simple semigroup · Least idempotent · E-Inversive semigroup

1 Preliminaries

Let S be a semigroup and $a \in S$. An element $x \in S$ is called a *weak inverse* of a if $xax = x$; the set of all weak inverses of a is denoted by $W_S(a)$. A semigroup S is said to be *E-inversive* if for every $a \in S$ there is $x \in S$ such that $ax \in E_S$, where E_S (or briefly E) is the set of idempotents of S (more generally, if $A \subseteq S$, then E_A denotes the set of idempotents of A). If $A \subseteq S$, then by A^* we shall mean the set of all non-zero elements of A. Since each semigroup with zero is E-inversive, then we define a semigroup S with zero to be *E-inversive* if for all $a \in S^*$ there exists $x \in S$ such that $ax \in E_S^*$. Finally, put $W_S^*(a) = W_S(a) \setminus \{0\}$ ($a \in S$). Recall from [3] that a semigroup S [with zero] is $E^{[*]}$-inversive if and only if $W_S^{[*]}(a) \neq \emptyset$ for every $a \in S^{[*]}$.

Lemma 1.1 *A semigroup S* [*with zero*] *is $E^{[*]}$-inversive if and only if every* [*non-zero*] *ideal of S contains some* [*non-zero*] *idempotent of S.*

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Proof Suppose that every non-zero ideal of S contains some non-zero idempotent of S, $a \in S^*$. Then S^1aS^1 contains at least one non-zero idempotent of S, that is, $xay = e$ for some $x, y \in S^1$ (in fact, we may suppose that $x, y \in S$), $e \in E_S^*$. Hence $exaye = e$, so $(yex)a(yex) = yex \neq 0$; otherwise $0 = xa(yex) = (xay)ex = ex$. Thus $0 = exay = e$, a contradiction. Consequently, $yex \in W_S^*(a)$.

The converse implication is clear. □

Lemma 1.2 *Let S be an $E^{[*]}$-inversive semigroup. Then eSe is $E^{[*]}$-inversive for every $e \in E_S^{[*]}$.*

Proof Observe first that $e \in eSe$, so $eSe \neq \{0\}$. Let $a \in (eSe)^*$ and $x \in W_S^*(a)$. Then $x = xax = x(eae)x$. Hence $exe = (exe)a(exe)$. Furthermore, if $exe = 0$, then we get $xe = [(xe)a(ex)]e = (xea)(exe) = 0$, so $x = (xe)a(ex) = 0$, a contradiction. Thus $exe \in W_{eSe}^*(a)$, as exactly required. □

We say that a semigroup S is a *semilattice* if $a^2 = a$, $ab = ba$ for all $a, b \in S$. Further, a congruence ρ on a semigroup S is called a *semilattice* congruence if S/ρ is a semilattice. It is clear that the least semilattice congruence η on an arbitrary semigroup exists. Finally, a semigroup is said to be *η-simple* if $\eta = S \times S$.

The next lemma follows immediately from the Second Isomorphism Theorem.

Lemma 1.3 *A homomorphic image of an η-simple semigroup is η-simple.*

Let S be a semigroup. Recall that the *natural partial order* is the relation $\leq$, defined on E_S by $e \leq f$ if $e = ef = fe$. We say that a semigroup S (without zero) has a least idempotent e if $e \leq f$ for every $f \in E_S$. Note that If S has a zero, say 0, then clearly 0 is the least element of E_S with respect to $\leq$, but in such a case, we may say that S has a *least non-zero idempotent* if E_S^* contains the least element with respect to the natural partial order.

Let A be an ideal of a semigroup S. We say that S is an *ideal extension* of the semigroup A by the semigroup T if the Rees semigroup S/A is isomorphic to T. Finally, an ideal P of a semigroup S is called *prime* if the condition $ab \in P$ implies that $a \in P$ or $b \in P$ for all $a, b \in S$.

2 The main results

Remark that by Corollary 3.9 of [4], a semigroup S is η-simple if and only if S has no proper prime ideals.

Proposition 2.1 *Let $S \neq S^0$ be an η-simple semigroup with a least idempotent. Then S is E-inversive. Moreover, S is an ideal extension of a group by an η-simple semigroup.*

Proof Let e be the least element of E_S. Then every ideal of S must contain e. Indeed, suppose by way of contradiction that there is an ideal A of S such that $e \notin A$. Let B

be the set theoretic union of all such ideals A of S. Then clearly B is the largest ideal of S such that $e \notin B$. Next, consider the Rees quotient S/B. Notice that we may think about S/B as a semigroup with zero, where all products not falling in S/B are zero. Consider now an arbitrary non-zero ideal C of S/B. Then by construction of B, $\{e\}$ must belong to C. Hence the intersection of all non-zero ideals of S/B contains $\{e\}$. In particular, S/B is E^*-inversive (see Lemma 1.1). Also, B is a prime ideal of S. Indeed, let $a, b \notin B$ be such that $ab \in B$. Then $fg \in B$ for some $f, g \in E_S \setminus B$ (because S/B is E^*-inversive). Hence $e = efg \in B$ (which is a contradiction). It follows that S has a proper semilattice congruence (by the above remark), a contradiction with the assumption of the theorem. Consequently, every ideal of S must contain e. Thus S has a kernel G (say) and S is E-inversive (Lemma 1.1). Hence for every $a \in S$ there exists $x \in S$ such that $ax, xa \in E_S$. Therefore $e = (ax)e = a(xe) \in aS$. We may equally well show that $e \in Sa$. It follows easily that S contains both a minimum left ideal L (say) and a minimum right ideal R (say). Furthermore, for every $a \in S$, La is a minimal left ideal of S (see [1], Lemma 2.32). Hence $La = L$, so L is an ideal of S (and $L = L^2$). We can show in a similar way that R is an ideal of S, so $L = R = G = eS = Se$ (because $Se \subseteq L, eS \subseteq R$, since $e \in L, R$). Consequently, $G = eSe$. Indeed, evidently $eSe \subset SeS = G$. Also, $G = GG = eSSe \subset eSe$. By Lemma 1.2, G is an E-inversive monoid (with an identity element e). Moreover, if $f \in E_{eSe}$, then $fe = ef = f$ i.e. $f \leq e$. Thus $f = e$. Consequently, G is a group ideal of S and so S is an ideal extension of the group G by the semigroup S/G which is η-simple, by Lemma 1.3, as required. □

Lemma 2.2 *Let $S \neq S^0$ be an η-simple semigroup with the least idempotent e. Then $ea = ae$ for all $a \in S$.*

Proof Let $a \in S$. Then $ea, ae \in eSe = eS = Se$, where eSe is a group ideal of S (see the proof of Proposition 2.1). Hence $e \cdot ae = ae$, $ea \cdot e = ea$. Thus $ea = ae$. □

A congruence on a semigroup is called a *group* congruence if the quotient semigroup is a group.

Corollary 2.3 *Let $S \neq S^0$ be an η-simple semigroup with a least idempotent, say e. Then the mapping $s \to es$ of S onto the group eS is an epimorphism leaving the elements of eS fixed. Moreover, the congruence σ induced by this morphism, that is $\sigma = \{(a, b) \in S \times S : ea = eb\}$, is the least group congruence on S.*

Proof The first part of the result follows from Proposition 2.1 and Lemma 2.2. Further, if ρ is a group congruence on S, then clearly $(s, es) \in \rho$ for every $s \in S$. Hence $\sigma \subseteq \rho$. □

Remark 1 Notice that if a semigroup $S \neq S^0$ with the least idempotent e is η-simple, then $\rho_{eS} \cap \sigma = 1_S$ and so S is a subdirect product of an (E-inversive) η-simple semigroup S/eS (with zero) and the group eS.

Further, the converse of Proposition 2.1 is valid.

Theorem 2.4 *A semigroup S without zero is η-simple and has a least idempotent if and only if it is an (E-inversive) ideal extension of a group by an η-simple semi-group.*

Proof The direct part follows from Proposition 2.1.

Conversely, let G be a group ideal of S (with an identity e) and $a \in S$. Then $ea \in G$, say $ea = g$. It follows that $g^{-1}ea = e \in Sa$. We may equally well show that $e \in aS$, so e is the least idempotent of E_S. Further, if ρ is a semilattice congruence on S, then (according to the proof of Theorem 5 in [5]) $\rho \cap (G \times G) = G \times G$. It follows that $\rho_G \subset \rho$, where ρ_G is the Rees congruence on S modulo G. Hence there is an epimorphism of S/ρ_G onto S/ρ. In fact, this morphism induced on S/ρ_G a semi-lattice congruence. Since S/ρ_G is η-simple, then S/ρ must be trivial. Consequently, $\rho = S \times S$, as exactly required. □

Remark that if a semigroup S is a *left* [*right*] *group* (i.e. $S \times S = \mathcal{L}$ [$\mathcal{R}$]), then S is η-simple. Indeed, let S be a left [right] group. Then $S \times S = \mathcal{L}$ [$\mathcal{R}$] $\subseteq \mathcal{J} \subseteq \eta$.

Theorem 2.5 *A semigroup S without zero is η-simple and has an idempotent e such that $ef = e$ [$fe = e$] if and only if it is an (E-inversive) ideal extension of a left* [*right*] *group by an η-simple semigroup.*

Proof ($\Longrightarrow$). Let $ef = e$ for every $f \in E_S$. We may equally well show like above (see the proof of Proposition 2.1) that e belongs to every ideal of S. Hence S has a kernel, say K. In particular, S is E-inversive. It follows that $e \in Sa$ for every $a \in S$. Thus S contains a minimum left ideal L and $L = La$ for all $a \in S$ (so $L = L^2$). Therefore $K = Se$ is a left simple semigroup (by Theorem 2.35 [1]) and so K is a left group (by the dual of Theorem 1.27 [1]). Consequently, S is an ideal extension of the left group K by the semigroup S/K which is η-simple.

($\Longleftarrow$). Let K be a left group ideal of S, $e \in E_K$ and $a \in S$. Then $ea \in K$, say $ea = k$. It follows that $ek^{-1}ea = ek^{-1}k = e \in Sa$, where k^{-1} is some inverse of k in K (since E_K is a left zero semigroup). Hence if $f \in E_S$, then $e = sf$ for some $s \in S$. Thus $ef = e$. We have just shown that $ef = e$ for all $e \in E_K$, $f \in E_S$. Further, if ρ is a semilattice congruence on S, then $\rho \cap (K \times K) = K \times K$ (by the preceding remark) and so $\rho = S \times S$ (by the proof of Theorem 2.4). □

Corollary 2.6 *Let S be a simple semigroup. If S has an idempotent e such that $ef = e$ [$fe = e$] for every $f \in E_S$, then S is a left* [*right*] *group.*

Proof Indeed, in such a case, $\mathcal{J} = S \times S$. It is almost evident (and also well-known) that $\mathcal{J} \subseteq \eta$. Hence S is η-simple, so S contains a left [right] group ideal K. Thus $S = K$. □

Notice that if S is a completely simple semigroup (see [2], Sect. 3.2), then the Green's relation $\mathcal{H}$ is a band congruence on S (see Lemma III.2.4 in [2]). Further, every left [right] group S is completely simple and E_S is a left [right] zero semigroup. It follows, from the above, that if S is a left [right] group, then $S/\mathcal{H}$ is a left [right] zero semigroup.

A semigroup S is said to be *congruence-free* if it has exactly two congruences.

Proposition 2.7 *Let S be a congruence-free semigroup without zero. If S has an idempotent e such that $ef = e$ $[fe = e]$ for every $f \in E_S$, then S is a simple group.*

Proof Let $ef = e$ for every $f \in E_S$. Since S is congruence-free, then either η is the identity or the universal relation on S. In the former case, S is a semilattice, but then e is the zero of S, a contradiction with the assumption of the proposition. It follows that S is η-simple. By Theorem 2.5, S contains a left group ideal K. Hence S is itself a left group. From the above remark we conclude that either $\mathcal{H} = 1_S$ or $\mathcal{H} = S \times S$. In the former case, S must be a left zero semigroup. Since $|S| > 1$, then the partition $\{\{e\}, S \setminus \{e\}\}$ of S induced a proper congruence on S, a contradiction. Thus $\mathcal{H} = S \times S$, so $E_S = \{e\}$, since $\mathcal{H}$ separates idempotents of S. Consequently, S is a simple group. □

Next, consider a semigroup S with zero such that the set E_S^* contains a least idempotent, say e. Remark that $fg \neq 0$ for all $f, g \in E_S^*$ (in fact, if $e \in E_S^*$ has the property that $ef = e$ $[fe = e]$ for every $f \in E_S^*$, then also $gh \neq 0$ for all $g, h \in E_S^*$).

Since a semigroup with zero adjoined has a proper semilattice congruence, then we shall say that a semigroup with zero is *η^*-simple* if S has *at most* two semilattice congruences, namely: (i) $S \times S$ or (ii) the congruence induced by the partition $\{\{0\}, S^*\}$. Clearly, the partition $\{\{0\}, S^*\}$ of a semigroup S with zero induces a semilattice congruence on S if and only if S is a semigroup with zero adjoined.

Recall that a semigroup S with zero is called a 0-*group* if S^* is a group.

Theorem 2.8 *A semigroup S with zero is η^*-simple and has a least non-zero idempotent if and only if it is an E^*-inversive semigroup with zero adjoined (and so S^* is an E-inversive semigroup with a least idempotent) and it is an ideal extension of a 0-group by an η-simple semigroup.*

Proof ($\Longrightarrow$). Let e be a least non-zero idempotent of S. We can show that every non-zero ideal of S contains e (see the proof of Proposition 2.1 and the above remark). In particular, S is E^*-inversive (Lemma 1.1). Hence for every $a \in S^*$ there is x such that xa is a non-zero idempotent of S. Thus $e \in Sa$. We may equally well show that $e \in aS$. Next, if $a, b \in S^*$, then (by the above) $e = xa, e = by$ for some $x, y \in S$. Hence $e = x(ab)y$ and so $ab \in S^*$. Consequently, S has no proper zero divisors. Thus S^* is an E-inversive semigroup with a least idempotent e and so S^* is an ideal extension of a group G by an η-simple semigroup (Theorem 2.4). It follows that S is an ideal extension of a 0-group G^0 by an η-simple semigroup. Indeed, S/G^0 must have a proper zero divisor (otherwise G^0 is a non-zero prime ideal of S).

The opposite implication follows easily from the proof of Theorem 2.4. □

A non-zero [left [right]] ideal A of a semigroup S with zero is called 0-*minimum* if it is contained in every non-zero [left [right]] ideal of S.

Further, a semigroup S with zero is called *categorical* if $abc = 0$ implies that either $ab = 0$ or $bc = 0$ for all $a, b, c \in S$.

Finally, we have the following theorem.

Theorem 2.9 *Let S be a categorical semigroup (with zero). Then S is η^*-simple and has a non-zero idempotent e such that $ef = e$ $[fe = e]$ for every $f \in E_S^*$ if and only if it is an E^*-inversive semigroup with zero adjoined (and so S^* is an E-inversive semigroup with a least idempotent) and it is an ideal extension of a left [right] group with zero adjoined by an η-simple semigroup.*

Proof ($\Longrightarrow$). Let $ef = e$ for every $f \in E_S^*$. We may equally well show like above that e belongs to every non-zero ideal of S. Hence S has a 0-minimum ideal K. In particular, S is E^*-inversive. It follows that $e \in Sa$ for all $a \in S^*$. Thus S contains a 0-minimum left ideal L (and $L = Le$, so $L = L^2$). Therefore $K = Se$ is a left 0-simple semigroup (by Theorem 2.35 in [1]), so K^* is a left simple semigroup (Theorem 2.27 in [1]). Thus K^* is a left group (by the dual of Theorem 1.27 in [1]). Further, suppose that $ea = 0$ for some $a \in S$ and let $b \in S^*$. Then $e = sb$ for some $s \in S$. Hence $sba = 0$. Thus $ba = 0$ (since S is categorical), so $\{0, a\}$ is a left ideal of S. It follows that either $\{0, a\} = K$ or $a = 0$. Consequently, $ea \neq 0$ for all $a \in S^*$. Therefore $ab \neq 0$ for all $a, b \in S^*$. Indeed, if $ab = 0$ for some $a, b \in S^*$, then $eb = 0$, a contradiction from the above. We conclude that S^* is an E-inversive semigroup, so S^* is an ideal extension of the left group K^* by the semigroup S/K which is η-simple (Theorem 2.5). Hence S is an ideal extension of the left group K with zero adjoined by the semigroup S/K which is η-simple, since K is not a prime ideal of S.

The opposite implication follows from the proof of Theorem 2.5. □

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